Community Structure in Multi-Mode Networks: Applying an Eigenspectrum Approach

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**Abstract**

We combine the logic of multi-mode networks developed in Fararo and Doreian (1984) with Newman’s (2006) spectral partitioning of graphs into communities. The resulting generalization of spectral partitioning provides a simple, elegant, and useful tool for discovering the community structure of multi-mode graphs. We apply the generalized procedure to a published three-mode network and find that the results of the algorithm are consistent with existing substantive knowledge. We also report the results of extensive simulations, which reveal that the generalization becomes more effective as the networks become denser.

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1. Introduction

Discovering cohesive subgroups, cliques, modules, or communities in networks has a rich history in the computer and social sciences (Fielder, 1973; Pothen, Simon & Liou, 1990; Wasserman & Faust, 1994; Jackson, 2008), but has seen an explosion of development since Girvan and Newman (2002) brought this problem to the attention of the mathematics and statistical physics communities (Porter, Onnela & Mucha, 2009). Communities within networks refer to densely connected subsets of vertices or nodes within the network. Several approaches have been leveraged to optimize solutions to identify communities, including partitional clustering procedures (e.g., Porter, Mucha, Newman, & Friend, 2007), centrality-based procedures (e.g., Girvan & Newman, 2002), and k-clique-based procedures (e.g., Palla, Derenyi, Farkas & Vicsek, 2005), among others. We focus on Newman’s (2006) spectral partitioning approach to modularity maximization because it is simple, intuitive, and quite popular.

Specifically, we generalize spectral partitioning to multi-mode networks, or networks that consist of more than one type of vertex (e.g., persons, groups, and events), using the logic for multi-mode networks set forth by Fararo and Doreian (1984). On the one hand, there exist several applications of modularity-based community algorithms to two-mode networks (e.g., Zhang et al., 2008). However, these applications rely on projections linking vertices in one mode (e.g., people) to other vertices in that mode through their mutual relations in the second mode (e.g., committees) (see Breiger, 1974). That is, some applications transform two-mode networks of who-to-what into one-mode networks of who-to-whom, and then apply community finding algorithms to the projections. On the other hand, Barber (2007) generalized modularity-based methods for partitioning networks into communities to the realm of bipartite graphs or two-mode networks, and Guimerà, Sales-Pardo, and Amaral (2007) developed an algorithm for examining the community structure of one of the modes in a bipartite graph. Here, we generalize modularity-based methods to n-mode networks, and apply the resulting algorithm to a published example, illustrating the utility and accuracy of the procedure based on substantive and qualitative knowledge. We also apply the algorithm to large simulated networks, illustrating some properties associated with its efficiency.

Community detection in networks or graphs seeks to partition the vertices into communities within which there is a concentration of ties. In social networks there may be cliques or groups of friends within which many ties are shared, while relatively few ties are sent to the rest of the network. On the internet there are communities of web sites with related topics that share links at above average rates, and yet send few links outside their community. The task, then, of community detection algorithms is to determine a useful way to partition the network into communities. One metric that has been developed in this connection is modularity, which reflects the extent, relative to a null model, to which edges are found within communities instead of between communities. Modularity provides a benchmark for comparing possible partitions of the vertices in a network.

Unfortunately, there is no way to ensure that any modularity solution is the optimal solution; optimization of the community structure is known to be NP-hard, and several procedures, old and new, have been leveraged with respect to optimization (Porter et al., 2009). The procedure we focus on here for optimizing modularity is based on spectral partitioning of the so-called “modularity” matrix. Specifically, we use the eigenspectrum of the modularity matrix, which we adjust to account for multi-mode networks. Below we review modularity maximization based on the eigenspectrum. We next elaborate an appropriate means of applying the maximization to multi-mode networks, and we apply the algorithm to a published three-mode network. We then report the results from simulations of large four-mode networks that illustrate the utility and flexibility of our approach.

2. Modularity Maximization

The Newman algorithm based on the eigenspectrum of a network is elegantly simple. Consider a network with \( n \) vertices (nodes) and \( m \) edges (relations) defined by a binary symmetric adjacency matrix \( A \) where an edge is denoted by a ‘1’ and \( A_{ij} \) is ‘0’ otherwise. Let \( P \) denote a matrix under a null model such that the \( P_{ij} \) are probabilities in the null model that an edge exists between vertices \( i \) and \( j \). In this paper, and in most cases, the null model is a simple model of independence such that \( P_{ij} = P_{i'j'} \). Now the so-called modularity matrix (\( B \)) refers to the difference between \( A \) and \( P \):

\[
B_{ij} = A_{ij} - P_{ij}
\]  

Newman (2006) recommends using the eigenspectrum of \( B \) to partition the vertices into modules. Specifically the eigenvector associated with the leading eigenvalue of \( B \) partitions the vertices into an optimal two-community solution such that the vertices in one community will have positive (or zero) eigenvector scores and those in the other community will have negative scores (Newman, 2006). Subsequent splits of the vertices into more than two communities can be identified similarly by looking at the signs of the eigenvector associated with the second leading eigenvalue, and so on. In general, the upper bound on the number of communities that may be found in this way is equal to one plus the number of positive eigenvalues of \( B \) (Newman, 2006). The algorithm uses these splits to identify communities, but the preferred solution is the one that maximizes the modularity \( Q \). That is, the algorithm converges when any subsequent split of the community structure makes a zero or negative contribution to the modularity \( Q \).

Let \( c \) denote the number of modules or communities that a graph is to be partitioned into. Then let \( S \) denote an \( n \times c \) matrix where each row is an index vector indicating vertex \( i \)’s membership in module \( j \) by a ‘1’ with ‘0’ elsewhere in the row. Each node may be assigned to only one module, making the unit vectors orthogonal (Barber, 2007). With \( S \) thusly defined, the modularity is as follows:

\[
Q = 1 / (2m) \times Tr S^T BS
\]
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Where $T_r$ refers to the trace of the matrix product $S^T BS$ and $S^T$ refers to the transpose of matrix $S$ (Barber, 2007; Newman, 2006).\(^1\) Maximization of $Q$ yields the optimal community structure of the graph.\(^2\)

3. Multi-Mode Networks and Modularity

In addition to one-mode networks, bipartite networks have been a focus of some analyses of community structure (Barber, 2007; Zhang et al., 2008), but tripartite or quadripartite networks have received much less attention (c.f. Mucha et al., 2010; Murata, 2011, for exceptions). Bipartite networks have a much simpler structure than do generalized multi-mode networks because one mode constitutes the rows of the matrix while the other mode constitutes the columns. With three or more modes, adjustments are required to place the modes in the same adjacency space. Fararo and Doreian (1984; see also Carley, 2003) have shown how to compile multi-mode networks where the number of modes exceeds two. We review this strategy and discuss how to transform these multi-mode networks into modularity matrices amenable to spectral partitioning.

Three-mode networks (or $n^2$-mode networks, where $n$ is any positive integer) require a block off-diagonal form in order to put all of the modes into a single adjacency space. Of course, this assumes there are no within-mode ties. For example, assume a network of edges between three types of vertices: persons, committees, and organizations. Denote the matrix of membership of persons on committees by $C$, the matrix of persons’ affiliation with organizations by $D$, and the matrix linking committees to organizations by $E$. (We assume that some committees may draw members from multiple organizations, and that similar cross-cutting affiliations are possible with respect to all pairs of modes.) Then the block off-diagonal matrix representation of the three-mode network, denoted $Z$, may be represented as follows (Fararo & Doreian, 1984):

\[
Z = \begin{pmatrix}
0 & C & D \\
C^T & 0 & E \\
D^T & E^T & 0
\end{pmatrix}
\]  

(3)

Matrix $Z$ may not be directly transformed into a modularity matrix $B$ because that would violate that matrix $Z$ be block off-diagonal. Consequently, we propose to compute the null model separately for each of the two-mode matrices that constitutes the three-mode matrix, to subtract out the null from each two-mode matrix, and then to aggregate them into the full three-mode modularity matrix. Let $C_p$, $D_p$, and $E_p$ denote the two-mode adjacency matrices, let $C_p^T$, $D_p^T$, and $E_p^T$ represent the transpose of $C_p$, $D_p$, and $E_p$, respectively. Then the appropriate three-mode adjacency matrix on which to compute the modularity $Q$ is of the following form:

\[
Z_g = \begin{pmatrix}
0 & C_g & D_g \\
C_g^T & 0 & E_g \\
D_g^T & E_g^T & 0
\end{pmatrix}
\]  

(4)

Although $Z_g$ is a three-mode matrix, the logic of its compilation can be extended to any number of modes. West, Melamed, and Breiger (2012) have applied this procedure to finding communities in four-mode narrative networks of people, groups, events, and games. We now turn to an example using a published three-mode network.

4. Ecology of Games and Tripartite Networks

Cornwell, Curry and Schwirian (henceforth CCS; 2003) analyzed a three-mode network consisting of actors, issues, and games organized around a major conflict in an urban community: the construction of a large-scale sports stadium in Cincinnati, OH during the 1990’s. The three “modes” included five actors (some of whom were individuals, such as the general manager of Cincinnati’s football team, and some organizational, such as the City Council), nine issues (for example, creating a referendum on whether to build a new stadium), and six games (such as the territorial game and the sports franchise game).\(^3\) As the total number of nodes was only 20, the multi-mode network data collected by CCS provides an ideal didactic example for demonstrating the usefulness of our proposed procedure. CCS aimed to implement network techniques and procedures of analysis to formalize Long’s (1958) ecology of games perspective, where games refer broadly to agendas and the domains within which they are pursued. CCS used multi-dimensional scaling and variants of network density to identify key nodes and groupings of nodes, and argued that their approach to the ecology of games aids in understanding the structure and process of community affairs.

CCS also published their data (2003, p. 133), along with much qualitative information about the controversy and the networks that were implicated in it. The key players (actors) in this network are Mike Brown, the manager of the Cincinnati Bengals football team, Marge Schott, the owner of the Cincinnati Reds baseball team, the City Council, the County Commissioners, and the general public. The issues consist of the new facility (NewFacility) that would keep the Bengals in

\[\text{null models for matrices } C, D, \text{ and } E, \text{ and let } C_p, D_p \text{, and } E_p \text{ denote the modularity matrices for } C, D, \text{ and } E \text{ (i.e., } C_g = C_p, \text{ } D_g = D_p, \text{ } E_g = E_p \text{). Then the appropriate three-mode adjacency matrix on which to compute the modularity } Q \text{ is of the following form: }\]

\[
Z_g = \begin{pmatrix}
0 & C_g & D_g \\
C_g^T & 0 & E_g \\
D_g^T & E_g^T & 0
\end{pmatrix}
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Cincinnati, proposed renovations to the old stadium (ProposingStadiumRenovations), the creation of the referendum (CreatingReferendum) to build a new stadium, passing the referendum (PassingReferendum), where to build the new stadium (WhereToBuild), determining the preliminary site (PreliminarySite), actually drafting the terms of the deal (DraftingTerms), stalling the transfer of the land (LandTransfer), and reaching a transfer agreement on the land (ReachingAgreement). The games consist of politics (Politics), urban redevelopment (UrbanRedevelopment), sports franchises (SportsFranchise), business competition (BusinessCompetition), territory (Territory), and the budget (Budget). For more detail on any of these nodes, please see CCS (2003, pp. 128-132). The three-mode network is visually represented in Figure 1 with triangles denoting people, circles denoting issues, and squares denoting games.

Figure 1. Sociogram of three-mode network. Ellipses denote communities and shapes denote modes (triangles are persons, circles are issues, and squares are games).

To determine the community structure of the CCS network, we computed the modularity matrices for each of the two-mode matrices that constitute the full three mode matrix (i.e., the $C_{sr}$, $D_{sr}$, and $E_{sr}$ matrices from above), and computed its eigenstructure. The results of the algorithm suggest that a three-community solution results in the maximum modularity ($Q \approx .051$). Specifically, nodes that are positive on the leading eigenvector form one community, nodes that are negative on the leading eigenvector and positive on the second leading eigenvector form a second community, and nodes that are negative on both the first and second leading eigenvectors form a third community. The results of the algorithm are also presented in Figure 1, with communities denoted by ellipses. Figure 1 also shows that our optimal three-community solution combines actors, issues, and games within each of the identified communities.

CCS (2003, p. 135) present the results of a multidimensional scaling of a distance matrix derived from the three-mode network. Based on their substantive and qualitative knowledge of the network, they indicated two communities in the network, leaving a few nodes out of either community. The largest community they identify contains five issues (where to build, the preliminary site, drafting of the terms, the land transfer, and reaching an agreement), two players (the county commissioners and the city council) and three games (politics, the budget, and territory). Here we point out that our eigenspectrum approach to community finding identified the exact same community without any substantive knowledge. This is the community to the left in Figure 1. This community can generally be thought of as the community that formed around the logistics of building the new stadium. The second community (top right), consisting of Mike Brown, Marge Schott, business competition, renovations, and the new facility, which overlaps substantially with CCS’s second community, generally accounts for the business community and its interests. Finally, the third community (bottom right) accounts for the public side of the building of the new stadium, including nodes such as the general public, the sports franchise game, creating the referendum, and passing the referendum, which was subject to a public vote.

In this example, the results are quite consistent with the “picture” almost literally painted by CCS (2003) on the basis of their deep knowledge of the Cincinnati controversy. The communities that they infer overlap substantially with those that our algorithm identified, even though we made no inferences from substantive knowledge, but rather allowed the eigenspectrum of the three-mode modularity matrix to determine the partition. Having illustrated our approach with this didactic example, we now turn to the results of simulations using much larger networks.

5. Simulation Results

The results in this section are based on thousands of four-mode network simulations. Given the number of nodes in each mode, the density of the constituent two-mode networks, and the probability of a tie occurring within a community, we simulated each two-mode network, computed the null models for each network, subtracted the null from the simulated network, aggregated the six two-mode networks into a four-mode network (of the form $Z_{sr}$ from above), and then determined the community structure of the four-mode network based on maximization of the modularity $Q$ for the network. Below we report the proportion of times that modularity maximization identified the imposed community structure in the networks, but first we present more details of the simulation.

Ties in each of the two-mode networks that constitute the full four-mode network were allocated between two equally-sized communities with probability $p$ and $1 - p$. In the simulations we report here, the first mode had 50 nodes ($a$), the second mode had 100 nodes ($b$), the third mode had 150 nodes ($c$), and the fourth mode had 200 nodes ($d$). Thus in the $a \times b$ network, ties from the first half of the nodes in $a$ to the second half of the nodes in $b$ occurred with a probability of $1 - p$, as did the ties from the second half of the nodes in $a$ to the first half of the nodes in $b$.

Aside from two equally-sized communities, we imposed a few other constraints for the sake of parsimony. First, the density of each of the two-mode networks that went into the full four-mode network was constrained to be equal. Second, the degree of each node in the first mode was constrained to be equal to the density times the number of nodes in the second mode. Ties were probabilistically distributed, but that does not

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*An R workspace and script are available on the first-listed author’s website that replicates the results reported in this paper.*
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We manipulated the density of the constituent two-mode networks to be between .2 and .5 in 1 intervals. We also manipulated the probability of a tie occurring within communities to be between .65 and .80 in .05 intervals. For each combination of densities and probabilities, we simulated 1,000 matrices and retained the community structure associated with the maximized modularity \( Q \). Figure 2 illustrates the proportion of times that the community structure we found matched the \textit{a priori} partition based on the probability of within-community ties. Not surprisingly, as the probability of within-mode ties increases, so too does the proportion of times that modularity maximization identifies the ‘correct’ community structure. In sparser networks, the probability of within-community ties has a large impact on the precision of the algorithm. With network densities of .2, the probability of a within-mode tie of .65 lead to the identification of only one correct community structure, and the probability of a within-mode tie of .80 lead to the identification of 999 correct community structures.

Somewhat surprisingly, network density has a reasonably strong effect on the accuracy of the algorithm. In sparser networks the probability of a within-community tie has more of an effect on the precision of the algorithm than in denser networks. For within-community ties with a probability of .7, for example, the algorithm correctly identifies 38.4% of the simulated networks’ community structure when the constituent matrices have a density of .2, but it identifies 98.2% of the simulated networks’ community structure when the constituent matrices have a density of .4. Thus, based on our simulation results, it appears that the precision of the approach outlined above is affected by the overall strength of the community structure, and the density of the networks. In retrospect, networks with more ties to probabilistically allocate within communities should result in more accurate community identification because there is more information to exploit. Also, although we do not report the results of our other simulations here, they suggest that the patterns found in Figure 2 are roughly reliable for significantly larger networks.

6. Conclusion

We have combined the logic of multi-mode networks with modularity-based community finding using spectral partitioning of the modularity matrix. We illustrated how to construct the multi-mode network before computing its eigenstructure. We then applied this algorithm to a published example and showed the overlap of our results with CCS’s (2003) results that were based on substantial substantive knowledge. We also reported on the results of simulations of large four-mode networks, which illustrated the importance of network density for the identification of community structure.

Three points warrant further mention. First, the procedure described herein can be applied to any number of modes. We analyzed a three-mode network, simulated four-mode networks, and West et al. (2012) used this procedure with four-mode data. Second, it is possible that within-mode ties may be incorporated with between-mode ties in a manner similar to that described above. This may be accomplished, for example, by treating the one-mode network as another two-mode network in the construction of \textbf{Z} (i.e., maintaining the block off-diagonal form, but including within-mode ties and their transpose). Such a formulation would maintain the assumption of a symmetrical adjacency matrix, but would actually be a multi-level network. Subsequent sensitivity analyses will be required to validate whether this identifies the community structure of multi-level networks acceptably well.5

Third, there is no limit to the number of vertices that can be partitioned into communities using our approach. In our empirical example, there were only twenty nodes, enabling us to compare results to an extant substantively meaningful account (Cornwell et al., 2003). As the number of vertices in a network increases, the ability to obtain and process substantive and qualitative knowledge decreases, thus increasing the need for reliable quantitative procedures such as the one we have proposed here. In this vein, the results of our simulations show the reliability of our procedure, and suggest that network density is an important part of the community structure puzzle.

References


5 We thank an anonymous reviewer for bringing within-mode ties to our attention.


